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SHORTER COMMUNICATIONS

A RIEMANN–HILBERT PROBLEM FOR A HEAT CONDUCTION IN AN INFINITE PLATE WITH A RECTANGULAR HOLE

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NOMENCLATURE

$2b, 2h$,	side lengths of the rectangular hole;
k,	thermal conductivity of the plate;
T,	temperature;
q_x, q_y,	x and y components of heat flux vector;
$q_{x\infty}, q_{y\infty}$,	x and y components of heat flux vector at infinity;
q_{x0}, q_{y0},	heat flux distributions on the circumference of the rectangular hole;
ϕ,	complex temperature;
Φ,	complex heat flux;
ζ,	auxiliary plane;
W,	auxiliary function.

INTRODUCTION

To DETERMINE the temperature distribution in a body, which contains some cracks or faults of various shapes in it, has been recognized to be very important in relation to not only the fracture of the material due to the thermal stress but also the heat conduction problem in the body, and the temperature or heat flux distributions about the holes of various shapes in the infinite region have been treated by several authors. That is, the case of the circular hole is treated by Carslaw and Jaeger [1], the elliptic one by Tranter [2] and the lines of discontinuities by Sih [3] respectively.

Coats worked out the unsteady heat conduction in the infinite region with a rectangular hole [4]. But, his method is based on the numerical technique and the heat flux distribution has not been expressed in the closed form. Moreover, Sih considered the singularity of the heat flux at the edge of the crack [3], but any consideration may not have been made on the singularity of the heat flux at the corner of the rectangular hole in the infinite plate, up to this time.

Therefore, in this paper, an analysis for a two-dimensional steady heat flux distribution in an infinite plate with a rectangular hole is made as the Riemann–Hilbert problem [5], and then the heat flux distribution is expressed in the closed form. Particularly, the singularity of the heat flux at the corner of the rectangular hole is examined in detail.

ANALYSIS

A rectangular hole of the side lengths $2b$ and $2h$ respectively is contained in an infinite plate whose thermal conductivity is assumed to be constant, k, and the coordinate system is taken like Fig. 1. On the circumference of the rectangle, the heat flux distribution normal to it is assumed to be given arbitrarily, and the heat flux at the infinity is uniform, $q_{x\infty} + iq_{y\infty}$ where q_x and q_y denote the x and y components of the heat flux vector.

The heat flux components are related to the temperature by

$$q_x = -k\frac{\partial T}{\partial x}, \qquad q_y = -k\frac{\partial T}{\partial y}. \tag{1}$$

In case of the steady heat conduction, the temperature distribution is well known to be given by a single analytic function of the complex variable $Z = x + iy$ [1],

$$\phi(Z) = T(Z) + iV(Z), \tag{2}$$

where the functions T and V are related to each other by the Cauchy–Riemann relations. From equations (1) and (2), the heat flux distribution may be also given by an analytic function, $(Z) = -q_x(Z) + iq_y(Z)$, and the solution to the two-dimensional steady heat conduction problem may be reduced to find out the analytic function ϕ or Φ which satisfies the given boundary conditions.

Now, the outer region of the rectangle is mapped into the upper half ζ-plane, as shown in Fig. 1, by

$$Z = A\int_0^\zeta \frac{\sqrt{[(\zeta^2 - d^2)(\zeta^2 - a^2)]}}{(\zeta^2 + 1)^2}\,d\zeta - b, \tag{3}$$

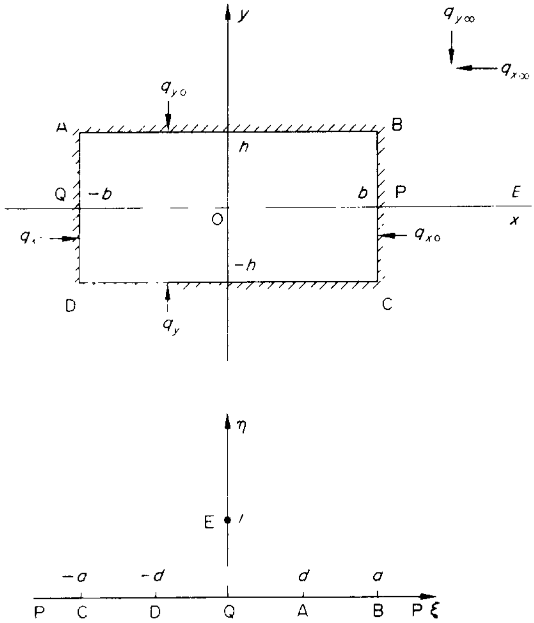

FIG. 1. Infinite plate with a rectangular hole and the auxiliary ζ-plane, $Z = x + iy$ and $\zeta = \xi + i\eta$.

where

$$\frac{b}{A} = \frac{i}{2}\int_{d}^{a} \frac{\sqrt{[(a^2 - \xi^2)(\xi^2 - d^2)]}}{(\xi^2 + 1)^2}\mathrm{d}\xi,$$

$$\frac{h}{A} = \frac{i}{2}\int_{-d}^{d} \frac{\sqrt{[(a^2 - \xi^2)(d^2 - \xi^2)]}}{(\xi^2 + 1)^2}\mathrm{d}\xi.$$

The boundary conditions on Φ in the ζ-plane are as follows,

$$\left.\begin{array}{lll} \mathrm{Re}\Phi(\xi) = -q_{x0}(\xi); & -\infty < \xi < -a \\ \mathrm{Im}\Phi(\xi) = \quad q_{y0}(\xi); & -a < \xi < -d \\ \mathrm{Re}\Phi(\xi) = -q_{x0}(\xi); & -d < \xi < d \\ \mathrm{Im}\Phi(\xi) = \quad q_{y0}(\xi); & d < \xi < a \\ \mathrm{Re}\Phi(\xi) = -q_{x0}(\xi); & a < \xi < \infty \\ \Phi(\zeta) = -q_{x\infty} + iq_{y\infty}; & \zeta = i \end{array}\right\} \quad (4)$$

Thus the problem is reduced into the Riemann–Hilbert problem [5]. Here, an auxiliary function W is introduced by

$$W(\zeta) = \sqrt{[(\zeta^2 - d^2)(\zeta^2 - a^2)]}\,\Phi(\zeta), \qquad (5)$$

in order that the real part of the function W is known on the entire real axis and then the well known Demtchencho's formula [6] is applicable to the present problem. Then the boundary conditions on W are, from equations (4) and (5),

$$\left.\begin{array}{ll} \mathrm{Re}W(\xi) = -\sqrt{[(\xi^2 - a^2)(\xi^2 - d^2)]}\,q_{x0}(\xi); & -\infty < \xi < -a \\ \mathrm{Re}W(\xi) = \quad \sqrt{[(a^2 - \xi^2)(\xi^2 - d^2)]}\,q_{y0}(\xi); & -a < \xi < -d \\ \mathrm{Re}W(\xi) = \quad \sqrt{[(a^2 - \xi^2)(d^2 - \xi^2)]}\,q_{x0}(\xi); & -d < \xi < d \\ \mathrm{Re}W(\xi) = -\sqrt{[(a^2 - \xi^2)(\xi^2 - d^2)]}\,q_{y0}(\xi); & d < \xi < a \\ \mathrm{Re}W(\xi) = -\sqrt{[(\xi^2 - a^2)(\xi^2 - d^2)]}\,q_{x0}(\xi); & a < \xi < \infty \end{array}\right\} \quad (6)$$

By the Demtchencho's formula, the solution for the present problem is obtained as,

$$W(\zeta) = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Re}W(\xi')}{\zeta - \xi'}\mathrm{d}\xi' + iP(\zeta), \qquad (7)$$

where P is a rational function of ζ with the real coefficients. Now, it is to be noticed that the acceptable singular points are limited to the four corners of the rectangle and Φ is bounded at the infinity, then the solution is given by

$$\Phi(\zeta) = \frac{i}{\pi\sqrt{[(\zeta^2 - a^2)(\zeta^2 - d^2)]}}\int_{-\infty}^{\infty} \frac{\mathrm{Re}\,W(\xi')}{\zeta - \xi'}\mathrm{d}\xi' + i\frac{C_0 + C_1\zeta + C_2\zeta^2}{\sqrt{[(\zeta^2 - a^2)(\zeta^2 - d^2)]}}, \qquad (8)$$

where $\mathrm{Re}\,W(\xi)$ is defined in equation (6). From the boundary condition at the infinity and also the condition of the unique temperature distribution in the plate, the unknown constants C_0, C_1 and C_2 are determined as follows,

$$\left.\begin{array}{l} C_0 = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \mathrm{Re}\,W(\xi')\frac{\xi'\,\mathrm{d}\xi'}{\xi'^2 + 1} + \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{Re}\,W(\xi')\frac{\xi'\,\mathrm{d}\xi'}{(\xi'^2 + 1)^2} \\ \qquad\qquad \dfrac{a^2 + d^2 + 2a^2d^2}{2\sqrt{[(a^2 + 1)(d^2 + 1)]}}q_{y\infty} \\ C_1 = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \mathrm{Re}\,W(\xi')\frac{\mathrm{d}\xi'}{\xi'^2 + 1} - \sqrt{[(a^2 + 1)(d^2 + 1)]}\,q_{x\infty} \\ C_2 = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \mathrm{Re}\,W(\xi')\frac{\xi'\,\mathrm{d}\xi'}{(\xi'^2 + 1)^2} + \frac{a^2 + d^2 + 2}{2\sqrt{[(a^2 + 1)(d^2 + 1)]}}q_{y\infty} \end{array}\right\} \quad (9)$$

It is clarified from equations (3) and (8) that the heat flux possesses the cube-root singularity in terms of the distance from the corner of the rectangular hole.

HEAT FLUX IN AN INFINITE PLATE WITH A CRACK

The solution obtained above presents that for arbitrary rectangle, and then it contains the solution for the crack as a special case. This special case is obtained as the limiting

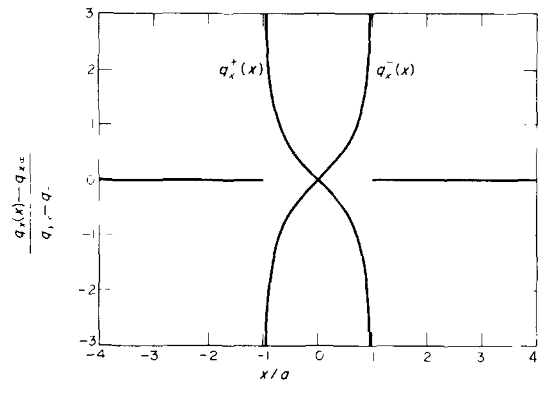

FIG. 2. Heat flux distribution on the x-axis in an infinite plate with a crack, $q_{y0}(x, +0) = q_{y0}(x, -0) = q_0 = \text{constant}$.

case $h \to 0$. The mapping function for this case, from equation (3), is

$$\zeta = i\sqrt{\left(\frac{Z+b}{Z-b}\right)}. \tag{10}$$

And from equations (8) and (9), the solution is

$$\Phi(\zeta) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{q_{y0}(\xi')}{\xi'-\zeta}\,d\xi' + \frac{C_0}{\zeta} + C_1 + C_2\zeta. \tag{11}$$

where

$$C_0 = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{q_{y0}(\xi')}{(\xi'^2+1)^2}\,d\xi' - \tfrac{1}{2}q_{y\infty},$$

$$C_1 = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{q_{y0}(\xi')}{\xi'^2+1}\,\xi'\,d\xi' - q_{x\infty},$$

$$C_2 = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{q_{y0}(\xi')}{(\xi'^2+1)^2}\,\xi'^2\,d\xi' + \tfrac{1}{2}q_{y\infty}.$$

Those results show clearly that the heat flux possesses the square-root singularity in terms of the distance from the edge of the crack and it is different from the cube-root singularity in the rectangle. Two simple cases are shown in a concrete form as follows.

(i) $q_{y0}(x, +0) = q_{y0}(x, -0) = q_0 = \text{constant}$.

In this case, equations (10) and (11) may be reduced into

$$\Phi(Z) = i(q_{y\infty} - q_0)\frac{Z}{\sqrt{[Z^2-a^2]}} + iq_0 - q_{x\infty}. \tag{12}$$

This result exactly agrees with one by Sih [3]. Numerical results for this simple case are shown in Fig. 2 where the square-root singularity at the edge of the crack is clearly examined, and also the uniform heat flux component at the infinity $q_{x\infty}$ is not disturbed on the x-axis outside the crack by the existence of the crack.

(ii) $q_{y0}(x, +0) = -q_{y0}(x, -0) = q_0 = \text{constant and } q_{y\infty} = 0$.

In this special case, the terms C_0 and C_2 in equation (11), which produce the square-root singularity at the edge of the crack, vanish because of the function $q_{y0}(\zeta')$ being an odd function of the integral variable ξ', and the result is

$$\Phi(Z) = \frac{q_0}{\pi}\log\frac{Z-a}{Z+a} - q_{x\infty}. \tag{13}$$

The heat flux for this case possesses the logarithmic-singularity at the edge of the crack.

CONCLUSION

The two-dimensional steady heat flux distribution about the rectangular hole in the infinite plate is analyzed based on the Riemann–Hilbert problem and then the heat flux distribution is expressed in the closed form. Particularly, the singularity of the heat flux at the corner of the rectangle is examined and it is clarified that the heat flux possesses the cube-root singularity in terms of the distance from the corner of the rectangle.

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